Noncommutative Point Sources

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ABSTRACT

We construct a perturbative solution to classical noncommutative gauge theory on $\mathbb{R}^3$ minus the origin using the Groenewald-Moyal star product. The result describes a noncommutative point charge. Applying it to the quantum mechanics of the noncommutative hydrogen atom gives shifts in the $1S$ hyperfine splitting which are first order in the noncommutativity parameter.

1 Introduction

As position eigenstates do not occur in theories with space-space noncommutativity, there can be no intrinsic notion of a *point* source in noncommutative physics. It then is argued that point charges become smeared in noncommutative gauge theory.[1] Gaussian distributions having width equal to the noncommutativity scale were utilized to model noncommutative sources, and in particular, sources associated with noncommutative black holes.[2]

On the other hand, after going to a star product realization of the noncommutative algebra, it is possible to show that point sources persist; i.e., there exist nonvanishing solutions to the noncommutative free field equations on $\mathbb{R}^n$ minus point(s). In particular, using the Groenewald-Moyal star product, it is easy to construct a perturbative solution to noncommutative $U(1)$ gauge theory describing a static point charge. We do this in section 2 for $n = 3$. Only space-space components of the noncommutativity tensor affect the fields around the static point source. A magnetostatic potential is induced at first order in the noncommutativity tensor, while corrections to the electrostatic potential are induced at second order. These lowest order corrections are independent of the choice of star product. The solution is nontrivial in the sense that it is not obtained from a Seiberg-Witten map of the commutative Coulumb solution. The latter would instead induce a nonvanishing current density away from the point source.

There is some utility in applying the lowest order solution to the quantum mechanics of the noncommutative hydrogen atom. A debate in the literature concerns how to treat the nucleus in the noncommutative theory.[3],[4],[5] If both the electron and nucleus are treated on the same footing; i.e., as noncommutative particles, their relative coordinates commute leading to no noncommutative corrections to the Coulumb potential.[5] On the other hand, it was argued that the nucleus should be treated as a commutative object since QCD effects dominate over any noncommutative physics.[4] Corrections then result in the Lamb shifts due to the noncommutativity of just the electron. It may be difficult to answer the debate conclusively in the absence of a consistent theory of noncommutative quarks and gluons. A pragmatic approach would be to set bounds on the noncommutativity of the nucleus. We do this in section 3 by presuming the nucleus to be a noncommutative point charge in the sense described above. New shifts result in the hydrogen atom spectra at lowest order in the noncommutativity parameter, including in the $1S$ hyperfine splitting.

Concluding remarks are made in section 4.

2 Point sources in noncommutative electrodynamics

Here we find it helpful to work in terms of SI units, with $c = 1$ (but not $\hbar = 1$), where the noncommutative gauge coupling constant g_{SI} has nontrivial units. Assuming constant

noncommutativity $\theta^{\mu\nu} = -\theta^{\nu\mu}$, the $U(1)$ gauge field equations read

$$\partial^\mu F_{\mu\nu} - ig_{SI}[A^\mu, F_{\mu\nu}]_\star = J_\nu \ , \tag{1}$$

with

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig_{SI}[A_\mu, A_\nu]_\star \ , \tag{2}$$

$[\ ,\ ]_\star$ being the star commutator associated with the Groenewald-Moyal star

$$\star = \exp\left\{\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial_\mu}\ \overrightarrow{\partial_\nu}\right\} \tag{3}$$

We can perturbatively solve these equations starting from the commutative Coulumb solution in three spatial dimensions

$$A^{(0)}_\mu = -\frac{q}{4\pi\epsilon_0\ r}\delta_{\mu 0} \ , \tag{4}$$

$\mu, \nu, ... = 0, 1, 2, 3$. Here we included the permittivity constant ϵ_0 in the current, $J^{(0)}_\mu = (q/\epsilon_0)\delta_{\mu 0}\delta^3(x)$. From (1) it then follows that

$$g = \frac{g_{SI}q}{4\pi\epsilon_0} \tag{5}$$

is a dimensionless factor. Take (4) to be the zeroth order term in a Taylor expansion in $\theta^{\mu\nu}$

$$A_\mu \quad = \quad A^{(0)}_\mu + A^{(1)}_\mu + A^{(2)}_\mu + \cdots \tag{6}$$

Assume that the noncommutative current J_μ vanishes everywhere away from the origin at all orders in $\theta^{\mu\nu}$

$$J_\mu = 0 \ , \qquad r \neq 0 \tag{7}$$

Then (1) gives

$$\begin{aligned} \nabla^2 A^{(1)}_0 &= 0 \\ (\nabla^2 - \partial_0^2)A^{(1)}_i + \partial_0\partial_i A^{(1)}_0 - \frac{gq}{4\pi\epsilon_0}\,\frac{\theta^{ij}x_j}{r^6} &= 0 \ , \qquad r \neq 0 \ , \end{aligned} \tag{8}$$

$i, j, ... = 1, 2, 3$, after extracting the first order terms and applying the Coulumb gauge $\nabla \cdot \vec{A} = 0$. A static first order solution is

$$A^{(1)}_i = \frac{gq}{16\pi\epsilon_0}\,\frac{\theta^{ij}x_j}{r^4} \ , \tag{9}$$

with $A^{(1)}_0 = 0$. (9) satisfies the Coulumb gauge condition due to the antisymmetry of θ^{ij}, and implies the existence of a noncommutative magnetic field

$$B^{(1)}_i = \frac{1}{2}\epsilon_{ijk}F^{(1)}_{jk} = -\frac{gq}{16\pi\epsilon_0}\ \epsilon_{ijk}\left\{\frac{\theta^{jk}}{r^4} - 4\frac{\theta^{j\ell}x_\ell x_k}{r^6}\right\} \tag{10}$$

We call (9) the inhomogeneous solution. It falls off faster than a magnetic dipole potential

$$\frac{\epsilon_{ijk} m_j^{(1)} x_k}{r^3} \,, \tag{11}$$

yet it cannot be expressed in terms of a magnetic quadrupole potential

$$\frac{\mathcal{M}_{ijk}^{(1)} x_j x_x}{2r^5} \,, \tag{12}$$

with constant coefficients $\mathcal{M}_{ijk}^{(1)}$. On the other hand, (11) and (12), along with higher moment potentials, can be regarded as homogeneous terms which can be added to (9). The moments are arbitrary, except for being linear in $\theta^{\mu\nu}$. (For instance, one can have $m_i^{(1)} \propto \theta^{i0}$ or $\epsilon_{ijk}\theta^{jk}$.) Additional homogeneous terms can be introduced with a multi-moment expansion for the time component of $A_\mu^{(1)}$:

$$A_0^{(1)} = -\frac{1}{4\pi\epsilon_0}\left\{\frac{q^{(1)}}{r} + \frac{p_i^{(1)} x_i}{r^3} + \frac{Q_{ij}^{(1)} x_i x_j}{2r^5} + \cdots\right\} \,, \tag{13}$$

where the constant coefficients $q^{(1)}$, $p_i^{(1)}$, $Q_{ij}^{(1)}$, ... are undetermined, except that they are linear in $\theta^{\mu\nu}$.

The first order solution (9) can be re-expressed in terms of the zeroth order solution (4) and its derivatives:

$$A_i^{(1)} = -\frac{1}{4} g_{SI} \theta^{ij} A_0^{(0)} \partial_j A_0^{(0)} \tag{14}$$

This is not a Seiberg-Witten map[6] of $A_0^{(0)}$, as commutative gauge transformations $A_0^{(0)} \to A_0^{(0)} + \partial_0\lambda$ do not induce noncommutative gauge transformations in A_μ. The standard expression for the Seiberg-Witten map at first order*

$$A_\mu^{(0)} \to A_\mu^{SW} \;=\; A_\mu^{(0)} \;-\; \frac{1}{2} g_{SI} \theta^{\rho\sigma}\, A_\rho^{(0)}\, (\partial_\mu A_\sigma^{(0)} - 2\partial_\sigma A_\mu^{(0)}) + \cdots \tag{15}$$

instead leads to a nonvanishing first order current density away from the origin (in addition to a singular current density at the origin). Substituting (9) in (15) gives

$$A_\mu^{SW} = -\frac{q}{4\pi\epsilon_0\, r}\Big(1 + g\frac{\theta^{0i} x_i}{r^3} + \cdots\Big)\delta_{\mu 0} \,, \tag{16}$$

which is associated with a nonvanishing current density for $r \neq 0$

$$J_0^{SW} = -\frac{qg}{\pi\epsilon_0}\,\frac{\theta^{0i} x_i}{r^6} + \cdots \qquad J_i^{SW} = -\frac{qg}{4\pi\epsilon_0}\,\frac{\theta^{ij} x_j}{r^6} + \cdots \,, \qquad r \neq 0 \tag{17}$$

It is straightforward to extend the inhomogeneous solution to higher orders. At second order in $\theta^{\mu\nu}$ the field equation (1) gives

$$\nabla^2 A_0^{(2)} + \frac{qg^2}{8\pi\epsilon_0}\left\{\frac{\mathrm{Tr}\theta^2}{r^7} - 7\frac{[\theta^2]^{ij} x_i x_j}{r^9}\right\} \;=\; 0$$

*Homogeneous terms $\mathcal{H}_{\mathcal{A}_\mu^{(0)}}$, satisfying $\mathcal{H}_{\mathcal{A}_\mu^{(0)}+\partial_\mu\lambda} - \mathcal{H}_{\mathcal{A}_\mu^{(0)}} \;=\; \theta^{\rho\sigma}\partial_\rho\lambda\partial_\sigma\mathcal{H}_{\mathcal{A}_\mu^{(0)}}$ at first order, can be added to this expression.[7],[8],[9]

$$(\nabla^2 - \partial_0^2)A_i^{(2)} + \partial_0\partial_i A_0^{(2)} \;=\; 0\;, \qquad r \neq 0\;, \tag{18}$$

in the Coulumb gauge. It is solved by

$$A_0^{(2)} = -\frac{qg^2}{16\pi\epsilon_0}\left\{\frac{\mathrm{Tr}\theta^2}{5r^5} - \frac{[\theta^2]^{ij}x_i x_j}{r^7}\right\}\;, \tag{19}$$

and $A_i^{(2)} = 0$. $A_0^{(2)}$ then falls off faster than an electric quadrupole potential, but cannot be expressed as an octopole potential.

The lowest order corrections to the Coulumb potential, (9) and (19), were computed using the leading order of the star commutator. They are therefore independent of the choice of star product.

3 Another look at the noncommutative hydrogen atom

Now consider a 'noncommutative' electron moving in the potential found in the previous section. Following [3] its quantum algebra is defined by

$$\begin{aligned}
[\hat{x}_i, \hat{x}_j] &= i\theta_{ij} \\
[\hat{x}_i, \hat{p}_j] &= i\hbar\delta_{ij} \\
[\hat{p}_i, \hat{p}_j] &= 0\;,
\end{aligned} \tag{20}$$

along with the usual spin algebra. It is well known that this can be mapped to the standard Heisenberg algebra, spanned by $\hat{X}_i$ and $\hat{P}_j$, using

$$\hat{x}_i \rightarrow \hat{X}_i = \hat{x}_i + \frac{1}{2\hbar}\theta_{ij}\hat{p}_j \qquad \hat{p}_i \rightarrow \hat{P}_i = \hat{p}_i \tag{21}$$

For the dynamics in a noncommutative gauge field we can adapt the standard Hamiltonian for a nonrelativistic electron

$$\hat{H} = \frac{1}{2m}\Big(\hat{p}_i - qA_i(\hat{x})\Big)^2 + qA_0(\hat{x}) - \frac{2\mu_B}{\hbar}\vec{S}\cdot\vec{B}(\hat{x})\;, \tag{22}$$

where $\mu_B = q\hbar/2m$. Alternatively, $\hat{H}$ can be realized in terms of differential operators acting on wavefunctions on $\mathbb{R}^3$, using the Groenewald-Moyal star (3). For example, the first term corresponds to

$$-\frac{\hbar^2}{2m}D_{\star i}D_{\star i} \tag{23}$$

The covariant derivative $D_{\star i}$ must be the same as that entering in the field equations (1) and the definition field strength (2), here written in the fundamental representation, i e.,

$$D_{\star i} = \partial_i - ig_{_{SI}}A_i\star \tag{24}$$

In comparing with (22) one gets the identification of g_{SI} with $q/\hbar$, or equivalently, the dimensionless coupling constant g defined in (5) with the fine structure constant:

$$g = \frac{q^2}{4\pi\epsilon_0\hbar} = \alpha \tag{25}$$

Next we substitute the solution for A_μ and $\vec{B}$ found in the previous section, keeping only the first order correction. The result is

$$\hat{H} = \frac{1}{2m}\left(\hat{p}_i - \frac{\alpha^2\hbar}{4}\,\frac{\theta^{ij}\hat{x}_j}{\hat{r}^4}\right)^2 - \frac{\alpha\hbar}{\hat{r}} + \frac{\alpha^2\hbar}{4m}\,\epsilon_{ijk}S_i\left\{\frac{\theta^{jk}}{\hat{r}^4} - 4\frac{\theta^{j\ell}\hat{x}_\ell\hat{x}_k}{\hat{r}^6}\right\}\,, \tag{26}$$

where $\hat{r}^2 = \hat{x}_i\hat{x}_i$. Since we only are interested in the first order in θ, it doesn't matter if we express the vector potential and magnetic field as functions of the commuting or noncommuting coordinates, $\hat{X}_i$ or $\hat{x}_i$. This of course is not the case for the Coulumb potential. Following [3], it can be re-expressed in terms of $\hat{X}_i$ using (21). Thus

$$\begin{aligned}
\hat{H} \;&=\; \hat{H}^{(0)} + \hat{H}_1^{(1)} + \hat{H}_2^{(1)} + \hat{H}_3^{(1)} + \cdots\,,\\
&\hat{H}^{(0)} = \frac{1}{2m}\hat{P}_i\hat{P}_i - \frac{\alpha\hbar}{\hat{R}}\\
&\hat{H}_1^{(1)} = -\,\frac{\alpha}{2}\,\frac{\vec{\theta}\cdot\vec{L}}{\hat{R}^3} \qquad\qquad \hat{H}_2^{(1)} = \frac{\alpha^2\hbar}{4m}\,\frac{\vec{\theta}\cdot\vec{L}}{\hat{R}^4}\\
&H_3^{(1)} = \frac{\alpha^2\hbar}{2m}\left\{\frac{2(\vec{X}\cdot\vec{S})(\vec{X}\cdot\vec{\theta})}{\hat{R}^6} - \frac{\vec{\theta}\cdot\vec{S}}{\hat{R}^4}\right\}\,,
\end{aligned} \tag{27}$$

where $\hat{R}^2 = \hat{X}_i\hat{X}_i$, $\theta_{ij} = \epsilon_{ijk}\theta_k$ and the dots indicate higher orders. $\hat{H}_1^{(1)}$ was obtained in [3], while $\hat{H}_2^{(1)}$ and $\hat{H}_3^{(1)}$ are the new corrections following from $A_i^{(1)}$, and are due to the noncommutativity of the source.† The latter contains couplings of the noncommutativity to both the orbital and spin angular momentum, respectively. Corrections to the Lamb shifts of the $\ell \neq 0$ states result from $\hat{H}_1^{(1)}$ and $\hat{H}_2^{(1)}$. The matrix elements are diagonalized by taking $\vec{\theta} = (0, 0, \theta)$. The former were computed in [3]. Similar expressions result for the latter. For the two $2P_{1/2}$ states:

$$< \hat{H}_1^{(1)} >_{2P_{1/2}^{\pm 1/2}} \;=\; -\,\frac{\alpha\theta}{2}\left\langle\frac{L_z}{\hat{R}^3}\right\rangle_{2P_{1/2}^{\pm 1/2}} = \mp\,\frac{\alpha\hbar\theta}{72a_0^3} \;=\; \mp\,\frac{\alpha^4 m\theta}{72\lambda_e^2} \tag{28}$$

$$< \hat{H}_2^{(1)} >_{2P_{1/2}^{\pm 1/2}} \;=\; \frac{\alpha^2\hbar\theta}{4m}\left\langle\frac{L_z}{\hat{R}^4}\right\rangle_{2P_{1/2}^{\pm 1/2}} = \pm\frac{\alpha^2\hbar^2\theta}{144ma_0^4} = \pm\,\frac{\alpha^6 m\theta}{144\lambda_e^2}\;, \tag{29}$$

using spectroscopic notation $n\ell_j^{m_j}$. The new contribution (29) is down by a factor of α^2 and thus gives a much weaker bound on θ. According to [10] the current theoretical accuracy on

†It was argued in [5] that the relative coordinate $\hat{x}_i$ is commuting, and that as a result the correction $\hat{H}_1^{(1)}$ to the Coulumb interaction is absent. On the other hand, the perturbations $\hat{H}_2^{(1)}$ and $\hat{H}_3^{(1)}$ persist when $\hat{x}_i$ is commuting, resulting in first order shifts in the hydrogen atom spectrum.

the $2P$ Lamb shift is about 0.08 kHz. From the splitting (28), this then gives the following bound on θ [‡]

$$\theta \lesssim (6\ \mathrm{GeV})^{-2}\ , \tag{30}$$

while from (29) one gets

$$\theta \lesssim (30\ \mathrm{MeV})^{-2} \tag{31}$$

As has been argued in [5], noncommutativity is not the same for all particles in noncommutative quantum mechanics. Here (30) is a bound on the noncommutativity associated with the test charge (electron), while (31) is effectively a bound on the lowest order noncommutativity of the source (proton). Comparing (31) with the QCD scale $\Lambda_{QCD} \sim 200$ MeV, one cannot here conclude that strong interactions dominate over any noncommutative effects of the source.

More interesting are the matrix elements of $\hat{H}_3^{(1)}$, as they induce new splittings in the $1S$ states, thus affecting the hyperfine structure.[§¶] Actually, with the restriction to static point sources, the $1S$ matrix elements are linearly divergent! To get a finite answer we take into account the finite size of the nucleus and insert the Λ_{QCD} cutoff[‖]

$$\begin{aligned} < \hat{H}_3^{(1)} >_{1S_{1/2}^{\pm 1/2}} &= \frac{\alpha^2\hbar}{2m} \left\langle \frac{S_i\theta_j}{\hat{R}^6}\,(2\hat{X}_i\hat{X}_j - \hat{R}^2\delta_{ij}) \right\rangle_{1S_{1/2}^{\pm 1/2}} \\ &= -\frac{\alpha^2\hbar\theta}{6m} \left\langle \frac{S_z}{\hat{R}^4} \right\rangle_{1S_{1/2}^{\pm 1/2}} = \mp \frac{\alpha^2\hbar\theta}{3ma_0^3\Lambda_{QCD}^{-1}} = \mp \frac{\alpha^5 m\theta}{3\hbar\lambda_e\Lambda_{QCD}^{-1}}\ , \end{aligned} \tag{32}$$

where again $\vec{\theta} = (0,0,\theta)$ and we used $\langle \hat{X}_i\hat{X}_j/\hat{R}^n\rangle_{\ell=0} = \frac{1}{3}\delta_{ij}\ \langle 1/\hat{R}^{n-2}\rangle_{\ell=0}$. These terms should then mix with the usual $1S$ hyperfine matrix elements. According to [10] the current theoretical accuracy on the $1S$ shift is about 14 kHz. From the splitting (32), this gives

$$\theta \lesssim (4\ \mathrm{GeV})^{-2}\ , \tag{33}$$

for the noncommutativity of the proton, which is now well above the QCD scale. However, without have a treatment of noncommutative QCD, the insertion of the QCD cutoff in this approach remains uncertain.

4 Concluding Remarks

We have found that the noncommutativity of the electron and proton have distinct experimental signatures in the hydrogen spectrum. We further found the same order of magnitude for

[‡] There was a computational error in the original version of [3].

[§] The $\ell = 0$ matrix elements would vanish with the addition of a term to $\hat{H}_3^{(1)}$ whereby the factor 2 in braces is changed to 3. The origin of such a term however is unclear.

[¶] Noncommutative corrections to the $1S$ hyperfine splitting were examined previously in [11] by expressing the dipole-dipole interaction in terms of the noncommutative coordinates $\hat{x}_i$. Those corrections, however, go like θ^2 at the lowest order.

[‖] This of course would not be valid for the muonium atom ($e^-\mu^+$). Relaxing the assumption of static sources, thereby taking into account recoil effects, may cure the ultraviolet divergence for that case.

their bounds.

There appear to be a number of possibilities for generalizations of this work: a) One is to obtain the exact solution for the noncommutative potential and also its dependence on the choice of star product. b) Another is to drop the restriction of static sources. This will allow for the study of recoil effects in noncommutative quantum systems. As stated earlier, this appears necessary to remove the divergence in the correction to the $1S$ state of the noncommutative muonium atom. c) A self-consistent dynamics for these point sources, at the classical as well as the quantum level, is then also of interest. The classical equations of motion would be analogous to the Wong equations in Yang-Mills theory.[12],[13] d) Generalizations to other gauge theories, including gravity, should be possible. For the case of gravity this should lead to yet another description of noncommutative black holes.[14] e) More challenging perhaps would be an attempt to find analogous solutions in theories with nonconstant noncommutativity.

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